### Quantum Subcritical Bubbles

Tomoko Uesugi and Masahiro Morikawa Department of Physics, Ochanomizu University, Tokyo 112, Japan

Tetsuya Shiromizu

Department of Physics, The University of Tokyo, Tokyo 113, Japan
and

Research Center for the Early Universe(RESCEU),
The University of Tokyo, Tokyo 113, Japan

#### Abstract

We quantize subcritical bubbles which are formed in the weakly first order phase transition. We find that the typical size of the thermal fluctuation reduces in the quantum-statistical physics. We estimate the typical size and the amplitude of thermal fluctuations near the critical temperature in the electroweak phase transition using quantum statistical average. Furthermore based on our study, we give implication on the dynamics of phase transition.

# 1 Introduction

The scenario of baryogenesis in the electroweak scale [1] is attractive because we can get the data of this energy scale from recent experiments of elementary particle physics. However, when one trys to explain the present baryon to entropy ratio in the minimal standard model, some serious problems exist. 1)In order to generate sufficient baryon number, we need a deviation from thermal equilibrium. This can be solved if the phase transition is strongly first order with supercooling. However, in the minimal standard model, electroweak phase transition is estimated to be weakly first order in the 1-loop approximation [2]. 2) CP-violation source from CKM phase is too small to explain present baryon to entropy ratio. 3) We could not avoid washing out of baryon number by the sphaleron transition after the phase transition in the minimal standard model. Improvements and more precise analysis are now in progress to solve the above problems. In this paper, we devote to the analysis of the phase transition. To understand the dynamics of the electroweak phase transition, one may need to consider a behavior of the thermal fluctuation of the scalar field. We study on a thermal fluctuation using subcritical bubbles which was first suggested by Gleiser et al[4]. In these analysis bubbles are treated as the classical object and assumed that the shape is spherical.

Recently one of the present authors insisted that the fluctuation of the subcritical bubble should be treated quantum mechanically. So one must quantize the bubble and consider up to higher energy levels. In this situation, one should take into account the modification from the O(3)-symmetry of the bubble because the energy difference between

each level is almost the same as differential energy. Another reason that we need to take into account the above modification exists. In our previous studies we have convinced that the phase mixing is sufficiently attained in the experimentally allowed region on the minimal standard model. From the above reasons, the modification cannot be neglected.

This paper is organized as follows. In Sec. II we summarize our previous studies. In Sec. III we quantize the collective coordinate of the subcritical bubble including the deviation form O(3)-symmetry and estimate the quantum-statistical averaged radius of the bubble. In Sec. IV we calculate the field amplitude inside the bubble. In the final section, we summarize our study.

# 2 The typical size of classical subcritical bubbles

We summarize in this section our previous results on the classical bubbles. In the weakly first order phase transition, the time development of the phase transition strongly depends on the thermal fluctuation. In the previous study [3], we have assumed

$$\phi(\mathbf{x}) = \phi_{+} \exp\left[-\frac{r^2}{R^2(t)}\right] \tag{1}$$

for the spatial profile of the thermal fluctuation.  $\phi_+$  is the field value at the asymmetric vacuum phase. The Hamiltonian for the radius R(t) becomes

$$H(P,R) = \frac{P^2}{2M(R)} + \frac{2}{5}M(R),\tag{2}$$

where

$$M(R) = \frac{15\pi^{3/2}\phi_+^2 R}{8\sqrt{2}}. (3)$$

This profile will be justified near the critical temperature at which two vacua degenerate because almost all bubbles have thick walls. Here we neglected the term from the potential of the scalar field because that term vanishes near the critical temperature. Thus we obtained the typical size of thermal fluctuation by thermally averaging the radius;

$$\langle R \rangle_T = \frac{\int dR dP \exp[-\beta H(P,R)]}{\int dR dP \exp[-\beta H(P,R)]} \simeq \frac{2\sqrt{2}T}{\pi^{3/2}.\phi_+^2}$$
(4)

Note that this is smaller than the correlation length. On the other hand, Gleiser et al. have assumed that the size of the bubble is equal to the correlation length[4]. We think that the typical size should be determined from the dynamics of the phase transition. However, we encounter with a serious problem: Can one treat the thermal fluctuation as the purely classical object? Actually, the number of state inside the bubble is too small;

$$n(r \le \langle R \rangle_T) \sim \mathcal{O}(1)$$
 (5)

Furthermore the condition for the decoherence, another measure for the system to be classical, is given by

$$\langle R \rangle_T \le R_c := \left(\frac{27\pi^{3/2}}{\sqrt{2}T\phi_+^2}\right)^{1/3} \sim 0.084 \text{GeV}^{-1}.$$
 (6)

On the other hand,  $\langle R \rangle_T \sim 0.012 \text{GeV}^{-1}$ , clearly,  $\langle R \rangle_T < R_c$ . Thus, one cannot expect that the subcritical bubble is a classical object.

From the above consideration one of the authors proposed that the fluctuation should be treated as quantum one with statistical fluctuation-dissipation in weakly first order phase transitions[5]. In this paper, we first treat the subcritical bubble as purely quantum and then we statistically average the quantum bubbles. If we obtained, after our present analysis, no drastic deviation from the previous estimate, we would conclude again, without solving the Master equation for the density matrix, that the phase transition cannot accompany the supercooling.

## 3 The typical size of quantum subcritical bubbles

#### 3.1 Subcritical bubble with modification

We notice that the energy difference is the same order of magnitude as the modification energy from the spherical symmetry in the statistical averaging. Hence, in advance, we consider the contribution of the modification from the spherical configuration. We express the spatial profile of the thermal fluctuation as

$$\phi(\mathbf{x}) = \sum_{\ell,m} \phi_{\ell,m}(r) Y_{\ell,m}(\Omega), \tag{7}$$

where  $Y_{\ell,m}(\Omega)$  is the spherical harmonics. Further, we assume the thick wall

$$\phi_{\ell,m}(r) = \phi_{+} \exp\left[-\frac{r^2}{R^2(t)}\right] \tag{8}$$

for each components. Here  $\phi_+$  denotes the vacuum expectation value of the scalar field in the broken phase. The Hamiltonian becomes

$$H(P,R) = H_0 + \sum_{\ell=1}^{\infty} (H_{\ell} - H_0), \tag{9}$$

where

$$H_{\ell}(P,R) = \frac{P^2}{2M(R)} + \frac{2}{5}M(R) + \frac{16}{15}\ell(\ell+1)M(R). \tag{10}$$

### 3.2 The quantization and the measure problem

Let us quantize the above system. Expressing the arbitrariness in the operator ordering by the parameter b, we write the quantum Hamiltonian as follows;

$$\hat{H}_{\ell} = -\frac{1}{aR} \frac{\partial^2}{\partial R^2} + \frac{1}{bR^2} \frac{\partial}{\partial R} + c_{\ell} R, \tag{11}$$

where a := 2M(R)/R and  $c_{\ell} := [(2/5) + (16/15)\ell(\ell+1)](M/R)$ .

In the following discussion, we set b/a = N and we concentrate on only two cases N = 1 and N = 2. Here one should be careful on the measure in the latter case. The former case N = 1 corresponds to the operator ordering;

$$\hat{H}_{\ell}(\hat{P}, R) = \hat{P} \frac{1}{2M(R)} \hat{P} + c_{\ell} R,$$
 (12)

where  $\hat{P} := -i\partial_R$ . Thus, the integral measure of this case is constant. The latter case N=2 corresponds to

$$\hat{H}_{\ell}(\hat{z}_p, z) = -\frac{1}{2}\hat{z}_p^2 + \cdots,$$
 (13)

where  $z := (R/R_0)^{3/2}$ ,  $R_0 := (R/M)^{1/2}$  and  $\hat{z}_p := -i\partial_z$ . Hence the integral measure is

$$\int_0^\infty dz \propto \int_0^\infty dR R^{1/2}.$$
 (14)

#### 3.3 The wave function

Let us construct the wave function of the subcritical bubble. The Schrödinger equation is given by

$$i\frac{\partial}{\partial t}\Psi_{\ell}(R,t) = \hat{H}_{\ell}\Psi_{\ell}(R,t). \tag{15}$$

For the stationary state ( $\Psi_{\ell} = \psi_{\ell} e^{-iE_{\ell}t}$ ,  $\hat{H}_{\ell}\psi_{\ell} = E_{\ell}\psi_{\ell}$ ), the Schrödinger equation becomes

$$-X_{\ell}'' + \left[ \frac{1}{2N} (\frac{1}{2N} + 1) x^{-2} + d_{\ell} \left( x - \frac{\epsilon_{\ell}}{2d_{\ell}} \right)^{2} \right] X_{\ell} = \frac{\epsilon_{\ell}^{2}}{4d_{\ell}} X_{\ell}, \tag{16}$$

where we have defined

$$X_{\ell} := R^{-\frac{1}{2N}} \psi_{\ell}(R)$$
 and  $x := a^{1/2} R$  (17)

and

$$d_{\ell} := \frac{c_{\ell}}{a} = \frac{1}{5} + \frac{8}{15}\ell(\ell+1), \quad \epsilon_{\ell} = \frac{E_{\ell}}{a^{1/2}} = \left(\frac{R}{2M}\right)^{1/2} E_{\ell}$$
 (18)

In order to solve this equation, we take the approximation,

$$x \gg \frac{\epsilon_{\ell}}{2d_{\ell}} \tag{19}$$

and we neglect the first term of the potential. Then the Schrödinger equation reduces to

$$-X_{\ell}'' + d_{\ell} \left( x - \frac{\epsilon_{\ell}}{2d_{\ell}} \right)^{2} X_{\ell} = \frac{\epsilon_{\ell}^{2}}{4d_{\ell}} X_{\ell}. \tag{20}$$

This approximation will be examined in the later discussion.

Now, we define

$$\xi := d_{\ell}^{1/4} \left( x - \frac{\epsilon_{\ell}}{2d_{\ell}} \right) \tag{21}$$

and then eq. (20) becomes

$$-\frac{d^2X_{\ell}}{d\xi^2} + \xi^2 X_{\ell} = \frac{\epsilon_{\ell}^2}{4d_{\ell}^{3/2}} X_{\ell}.$$
 (22)

As this equation is the same as that for the harmonic oscillator, the eigenfunctions and eigenvalues of this equation are given by

$$X_{\ell,n}(\xi) \propto (-1)^n e^{\frac{\xi^2}{2}} \frac{d^n e^{-\xi^2}}{d\xi^n},$$
 (23)

and

$$\frac{\epsilon_{\ell,n}^2}{4d_{\ell}^{3/2}} = 2n + 1,\tag{24}$$

respectively.

Using the above eigenvalues, we examine the approximation used on the potential under the assumption of (19). The original potential before the approximation is

$$V_{\ell}(x) := \frac{1}{2N} \left( \frac{1}{2N} + 1 \right) x^{-2} + d_{\ell}(x - x_0)^2, \tag{25}$$

where  $x_0 := \epsilon_{\ell,n}/2d_{\ell}$ . Let us call the minimum value at which the first term coincides with the second term to be  $x_*$ .

In the case of N=1

$$x_* = \frac{x_0}{2} \left[ 1 + \sqrt{1 + \frac{2 \cdot 3^{1/2}}{2n+1}} \right]$$
 for  $n = 0, 1$ . (26)

This is larger than  $x_0$  and the approximation brakes down. On the other hand, for  $n \geq 2$ ,

$$x_* = \frac{x_0}{2} \left[ 1 - \sqrt{1 - \frac{2 \cdot 3^{1/2}}{2n+1}} \right] \tag{27}$$

is smaller than  $x_0$  and it means that the approximation is reasonable. In the case of N=2, in a similar way, we can show that the approximation is reasonable only for  $n \geq 1$ .

Thus, for low energy states, we have to use different approximation. The simplest remedy for that is as follows. We first expand the potential

$$V(x) \simeq V(\overline{x}) + \frac{1}{2}V''(\overline{x})(x - \overline{x})^2 + \cdots, \tag{28}$$

where  $\overline{x}$  is determined by  $V'(\overline{x}) = 0$ . Defining

$$\eta := \left[\frac{1}{2}V''(\overline{x})\right]^{1/4}(x - \overline{x}),\tag{29}$$

the Schrödinger equation for the stationary state becomes

$$-\frac{d^2 X_{\ell}}{d\eta^2} + \eta^2 X_{\ell} = \frac{1}{\left[\frac{1}{2} V''(\overline{x})\right]^{1/2}} \left[\frac{\epsilon_{\ell}^2}{4d_{\ell}} - V(\overline{x})\right] X_{\ell}.$$
 (30)

The solution is given by

$$X_{\ell,n}(\eta) \propto (-1)^n e^{\frac{\eta^2}{2}} \frac{d^n e^{-\eta^2}}{d\eta^n}$$
 (31)

and

$$\frac{1}{\left[\frac{1}{2}V''(\overline{x})\right]^{1/2}} \left[\frac{\epsilon_{\ell}^2}{4d_{\ell}} - V(\overline{x})\right] = 2n + 1. \tag{32}$$

Under the above two approximation, the wave function badly behaves at  $R \sim 0$ . This violates the hermiticity of the Hamiltonian. However, this bad behavior simply comes

from our approximation and should not be taken seriously. Actually we can easily see the true behavior near  $R \sim 0$ . At  $x \ll 1$ , the eq. (16) becomes

$$-\frac{d^2X_{\ell}}{dx^2} + \frac{1}{2N} \left(\frac{1}{2N} + 1\right) x^{-2} X_{\ell} \approx 0.$$
 (33)

Assuming  $X_{\ell} \propto x^n$ , this becomes

$$\left(n - 1 - \frac{1}{2N}\right)\left(n + \frac{1}{2N}\right) = 0.$$
(34)

Thus, the regular solution is  $X_{\ell} \propto x^{1+1/2N} \propto R^{1+1/2N}$ . The original wave function is given by  $\psi_{\ell}(R) \propto R^{1+1/N}$ . The first derivative is  $\psi'(R) \propto R^{1/N}$ . Therefore actually the Hamiltonian is a hermitian operator.

### 3.4 The typical size

Now, we can calculate the typical size by the quantum-statistical averaging;

$$\langle R \rangle_Q = \frac{\sum_{\ell,n} e^{-\beta E_{\ell,n}} \langle R \rangle_{\ell,n}^Q}{\sum_{\ell,n} e^{-\beta E_{\ell,n}}}$$
(35)

where

$$\langle R \rangle_{\ell,n}^{Q} = \frac{\int_0^\infty dR R^\alpha R |\psi_{\ell,n}(R)|^2}{\int_0^\infty dR R^\alpha |\psi_{\ell,n}(R)|^2}.$$
 (36)

 $R^{\alpha}$  being the integral measure; for N=1,  $\alpha=0$ , and for N=2,  $\alpha=1/2$ . The typical size of subcritical bubbles does depend on the integral measure, however after our approximation of the potential (19), the dependence on N disappears. Then, the effect of the integral measure only appears in the wave function with small n. Fortunately even for these wave functions, we can easily show that this effect to the typical size is small enough. We now estimate the typical size of a bubble using the effective potential in a finite temperature with the 1-loop approximation [2]. We use  $m_W=80.6 \,\mathrm{GeV}$ ,  $m_Z=91.2 \,\mathrm{GeV}$  and  $m_t=174 \,\mathrm{GeV}$  for the W-boson, the Z-boson and the top-quark masses. And we set the mass of Higgs;  $m_H=65 \,\mathrm{GeV}$  that is the lower limit from the experiment [6]. Accordingly,  $\phi_+=49.4 \,\mathrm{GeV}$  and the critical temperature:  $T_c=98.3 \,\mathrm{GeV}$ .

The result of our numerical calculation for the typical size is,

$$\langle R \rangle_Q \sim 0.021 \text{GeV}^{-1}$$
 (37)

near the critical temperature. This size is larger than that of thermal average:  $\langle R \rangle_T \sim 0.012 \text{GeV}^{-1}$ .

To clarify the difference between classical and quantum regime, we calculate the distribution function  $(W(R_c))$ ;

$$W(R_c) = \int_{-\infty}^{\infty} dP R_c^{\alpha} e^{-iR_{\Delta}P} W(P, R_c)$$

$$= \int_{-\infty}^{\infty} dP R_c^{\alpha} \int_{-\infty}^{\infty} dR_{\Delta} e^{-iR_{\Delta}P} \rho \left( R_c + \frac{1}{2} R_{\Delta}, R_c - \frac{1}{2} R_{\Delta} \right)$$

$$= R_c^{\alpha} \rho(R_c, R_c), \tag{38}$$

where  $\rho$  is the density matrix given by

$$\rho\left(R_c + \frac{1}{2}R_{\Delta}, R_c - \frac{1}{2}R_{\Delta}\right) = \rho(R, R') = \text{Tr}_{\ell, n}\left[e^{-\beta E_{\ell, n}} \psi_{\ell, n}^*(R') \psi_{\ell, n}(R)\right]. \tag{39}$$

Under the present approximations, the final form of the distribution function does not depend on the measure;

$$W(R_c) = R_c^{\alpha} \rho(R_c, R_c) = R_c \operatorname{Tr}_{\ell,n} \left[ e^{-\beta E_{\ell,n}} X_{\ell,n}^2(R) \right]. \tag{40}$$

We depicted profile of quantum distribution function W(R) (solid line) and the classical distribution function (broken line) in figure 1. The location of the peak of W(R) is larger than that of classical distribution function. This is because the fluctuations in quantum distribution function comes from quantum fluctuation as well as the statistical fluctuations which is expressed in the classical distribution function.

# 4 Fluctuation strength in fields

While, fixing R to the above averaged value, we can estimate the fluctuation in the field value  $\phi$ . Following the manner of previous studies [3] [4], we set

$$\phi_{lm} = A(t) \exp\left[-\frac{r^2}{\langle R \rangle_Q^2}\right]. \tag{41}$$

In this case the Schrödinger equation becomes

$$\left(-\frac{1}{2m}\frac{\partial^2}{\partial A^2} + f_{\ell}A^2\right)\psi_{\ell} = E_{\ell}\psi_{\ell},\tag{42}$$

where

$$f_{\ell} = \frac{m}{\langle R \rangle_Q^2} \left[ \frac{3}{4} + 2\ell(\ell+1) \right] , \quad m = \frac{\pi^{3/2} \langle R \rangle_Q^3}{\sqrt{2}}$$

$$\tag{43}$$

and the wave function becomes

$$\psi_{\ell,n}(a) \propto (-1)^n e^{\frac{a^2}{2}} \frac{d^n e^{-a^2}}{da^n}.$$
 (44)

where we define

$$a = \left(2mf_{\ell}\right)^{\frac{1}{4}}A. \tag{45}$$

The fluctuation of  $\phi$  can be calculated naively by

$$\sqrt{\langle \phi^2 \rangle} = \sqrt{\frac{\sum_{\ell,n} e^{-\beta E_{\ell,n}} \langle \phi^2 \rangle_{\ell,n}}{\sum_{\ell,n} e^{-\beta E_{\ell,n}}}},$$
(46)

where

$$\langle \phi^2 \rangle_{\ell,n} = \frac{\int dA A^2 \psi_{\ell,n}(A)}{\int dA \psi_{\ell,n}(A)}.$$
 (47)

The result of our numerical calculation becomes

$$\sqrt{\langle \phi \rangle^2} \sim 30.4 \text{GeV}.$$
 (48)

At the reflection point,  $\phi = \phi_* = 10.5 \text{GeV}$ . The fluctuation (48) well exceeds  $\phi_*$  near the symmetric vacuum. This fact means that the true and false phases are well mixed at the scale typical in the fluctuations above the critical temperature. Therefore the use of 1-loop approximation of the effective potential is no longer valid.

## 5 Summary

In this paper, we quantized the subcritical bubbles and using quantum statistics, we calculated the typical size of subcritical bubbles as well as the typical fluctuation of fields.

As a result of our calculations, we found that the typical size of subcritical bubbles is larger than that calculated by the classical thermal average, and that the size of the typical fluctuation of the field similarly well exceeds the first reflection point in our case. Therefore we may conclude that the phase transition cannot accompany supercooling. On the other hand, we must confess that the above study is quite qualitative. Furthermore, we could not justify whether the electroweak phase transition is first order or not as long as we use the 1-loop approximation for the effective potential, not effective action.

Finally, we must mention the assumptions used here. First, we have assumed the Gaussian distribution for the spatial profile of the thermal fluctuation as in the previous studies. However there is no dynamical justification on this Gaussian distribution. Second, in order to consider the deviation from the spherical symmetric configuration, we assumed that each component does not depend on the parameter  $(\ell, n)$ . We hope that we can justify the above assumptions in our quantitative study in the near future.

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### Figure Captions

